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A The Central Bank's Policy Problem

The central bank minimizes (5) subject to (1) and (2). The Lagrangian of this problem is given by

$$\begin{aligned}\mathcal{L} =& E_t \left\{ \sum_{s=0}^{\infty} \beta^s \left[\frac{1}{2}(\pi_{t+s}^2 + \omega_x x_{t+s}^2 + \omega_i i_{t+s}^2) \right.\right. \\ & + \kappa_{1|t+s} \left[x_{t+s} - \alpha x_{t+s+1} - (1-\alpha)\theta^2 x_{t+s-1} + \sigma^{-1} i_{t+s} \right. \\ & \left. - \sigma^{-1}\alpha\pi_{t+s+1} - \sigma^{-1}(1-\alpha)\theta^2\pi_{t+s-1} \right] \\ & \left.\left. + \kappa_{2|t+s} \left[\pi_{t+s} - \beta\alpha\pi_{t+s+1} - \beta(1-\alpha)\theta^2\pi_{t+s-1} - \lambda x_{t+s} \right] \right] \right\}. \end{aligned} \tag{A.1}$$

The related first-order conditions are given by

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \pi_{t+s}} :& E_t \left\{ \beta^s \{ \pi_{t+s} + \kappa_{2|t+s} \} + \beta^{s-1} \{ \kappa_{1|t+s-1}[-\sigma^{-1}\alpha] + \kappa_{2|t+s-1}[-\beta\alpha] \} \right. \\ & \left. + \beta^{s+1} \{ \kappa_{1|t+s+1}[-\sigma^{-1}(1-\alpha)\theta^2] + \kappa_{2|t+s+1}[-\beta(1-\alpha)\theta^2] \} \right\} \stackrel{!}{=} 0 \end{aligned} \tag{A.2}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_{t+s}} :& E_t \left\{ \beta^s \{ \omega_x x_{t+s} + \kappa_{1|t+s} + \kappa_{2|t+s}[-\lambda] \} + \beta^{s-1} \{ \kappa_{1|t+s-1}[-\alpha] \} \right. \\ & \left. + \beta^{s+1} \{ \kappa_{1|t+s+1}[-(1-\alpha)\theta^2] \} \right\} \stackrel{!}{=} 0 \end{aligned} \tag{A.3}$$

$$\frac{\partial \mathcal{L}}{\partial i_{t+s}} : E_t \left\{ \beta^s \{ \omega_i i_{t+s} + \kappa_{1|t+s}\sigma^{-1} \} \right\} \stackrel{!}{=} 0 \tag{A.4}$$

for each date $s \geq 0$ and initial conditions $\kappa_{1|-1} = \kappa_{2|-1} = 0$, given that the central bank employs a commitment to its optimality conditions from a timeless perspective. Thus, we can equivalently express (A.2) to (A.4) as

$$\begin{aligned} 0 =& \pi_t - \beta^{-1}\sigma^{-1}\alpha\kappa_{1|t-1} + \kappa_{2|t} - \alpha\kappa_{2|t-1} \\ & - \beta\sigma^{-1}(1-\alpha)\theta^2 E_t\kappa_{1|t+1} - \beta^2(1-\alpha)\theta^2 E_t\kappa_{2|t+1} \end{aligned} \tag{A.5}$$

$$\kappa_{2|t} = \frac{\omega_x}{\lambda} x_t + \frac{1}{\lambda}\kappa_{1|t} - \frac{\alpha}{\beta\lambda}\kappa_{1|t-1} - \frac{\beta}{\lambda}(1-\alpha)\theta^2 E_t\kappa_{1|t+1} \tag{A.6}$$

$$\kappa_{1|t} = -\sigma\omega_i i_t. \tag{A.7}$$

B Time-Consistency of Misspecification in Policy Design

DOUBTS ABOUT PARAMETRIZATION. Such doubts can be expressed via

$$x_t = \eta_1 E_t x_{t+1} - \eta_2 (i_t - E_t\pi_{t+1}), \quad \text{and} \tag{B.1}$$

$$\pi_t = \zeta_1 E_t\pi_{t+1} + \zeta_2 x_t, \tag{B.2}$$

where η_1, η_2, ζ_1, and ζ_2 are arbitrary coefficients.

In solving the central bank's problem, as outlined above, one arrives at a specific targeting rule

$$\pi_t = -\frac{\omega_x}{\zeta_2}(x_t - \beta^{-1}\zeta_1 x_{t-1}). \tag{B.3}$$

The expectations-based reaction function based on (B.1) to (B.3) yields reduced form

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\zeta_1\zeta_2}{\omega_x+\zeta_2^2} \\ 0 & \frac{\zeta_1\omega_x}{\omega_x+\zeta_2^2} \end{bmatrix} \begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} + \begin{bmatrix} \frac{\beta^{-1}\zeta_1\omega_x}{\omega_x+\zeta_2^2} & 0 \\ \frac{\beta^{-1}\zeta_1\zeta_2\omega_x}{\omega_x+\zeta_2^2} & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix}, \tag{B.4}$$

and states the central bank's *perceived law of motion* (PLM), which can be expressed as (8) with matrices $\mathbf{A}(\zeta_1,\zeta_2)$ and $\mathbf{C}(\zeta_1,\zeta_2)$ for convenience. Now, following the logic from above, the *actual law of motion* (ALM) results from an expectations-based reaction function achieved by combining (B.3) with (1) to (4). It leads to reduced form

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\beta\alpha\zeta_2}{\omega_x+\zeta_2\lambda} \\ 0 & \frac{\beta\alpha\omega_x}{\omega_x+\zeta_2\lambda} \end{bmatrix} \begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} + \begin{bmatrix} \frac{\beta^{-1}\zeta_1\omega_x}{\omega_x+\zeta_2\lambda} & -\frac{\zeta_2\beta(1-\alpha)\theta^2}{\omega_x+\zeta_2\lambda} \\ \frac{\beta^{-1}\zeta_1\lambda\omega_x}{\omega_x+\zeta_2\lambda} & \frac{\beta(1-\alpha)\theta^2\omega_x}{\omega_x+\zeta_2\lambda} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix}, \tag{B.5}$$

or

$$\mathbf{y}_t = \mathcal{T}_A(\zeta_2)\ E_t\mathbf{y}_{t+1} + \mathcal{T}_B(\zeta_1,\zeta_2)\ \mathbf{y}_{t-1}, \tag{B.6}$$

where $\mathcal{T}_A(\zeta_2)$ and $\mathcal{T}_B(\zeta_1,\zeta_2)$ state the $\mathcal{T}$-mapping from the PLM to the ALM.

The ALM being consistent with the PLM requires that in equilibrium

$$\begin{pmatrix} \mathcal{T}_A(\zeta_2) \\ \mathcal{T}_B(\zeta_1,\zeta_2) \end{pmatrix} = \begin{pmatrix} \mathbf{A}(\zeta_1,\zeta_2) \\ \mathbf{C}(\zeta_1,\zeta_2) \end{pmatrix}. \tag{B.7}$$

One can show that (B.7) fails to be satisfied. In other words, the policymaker cannot successfully pin down all the parameters in (B.3). In the logic of a *self-confirming equilibrium*, the policymaker's doubts about the parametrization of the model for policy design are justified, i.e., the policy is *time-inconsistent*.

Doubts About Variables Included. Alternatively, missing its target may motivate the policymaker to include lagged variables into the model for policy design, which can be expressed via

$$x_t = \eta_1 E_t x_{t+1} - \eta_2\left(i_t - E_t\pi_{t+1}\right) + \eta_3 x_{t-1} + \eta_4\pi_{t-1}, \quad \text{and} \tag{B.8}$$

$$\pi_t = \zeta_1 E_t\pi_{t+1} + \zeta_2 x_t + \zeta_3\pi_{t-1} \tag{B.9}$$

for arbitrary coefficients η_1, η_2, η_3, η_4, ζ_1, ζ_2, and ζ_3.

Following the very same steps as before, the specific targeting rule is given by

$$\pi_t = -\frac{\omega_x}{\zeta_2}(x_t - \beta^{-1}\zeta_1 x_{t-1} - \beta\zeta_3 E_t x_{t+1}). \tag{B.10}$$

An expectations-based reaction function based on (B.8) to (B.10) implies the PLM

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \frac{\beta\zeta_3\omega_x}{\omega_x+\zeta_2^2} & -\frac{\zeta_1\zeta_2}{\omega_x+\zeta_2^2} \\ \frac{\beta\zeta_2\zeta_3\omega_x}{\omega_x+\zeta_2^2} & \frac{\zeta_1\omega_x}{\omega_x+\zeta_2^2} \end{bmatrix} \begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} + \begin{bmatrix} \frac{\beta^{-1}\zeta_1\omega_x}{\omega_x+\zeta_2^2} & -\frac{\zeta_2\zeta_3}{\omega_x+\zeta_2^2} \\ \frac{\beta^{-1}\zeta_1\zeta_2\omega_x}{\omega_x+\zeta_2^2} & \frac{\zeta_3\omega_x}{\omega_x+\zeta_2^2} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix}. \tag{B.11}$$

Again, we can capture this PLM as (8) with matrices $\mathbf{A}(\zeta_1,\zeta_2,\zeta_3)$ and $\mathbf{C}(\zeta_1,\zeta_2,\zeta_3)$. On the other hand the ALM resulting from an expectations-based reaction function based on (B.10) and (1) to (4) is now given by

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \frac{\beta\zeta_3\omega_x}{\omega_x+\zeta_2\lambda} & -\frac{\beta\alpha\zeta_2}{\omega_x+\zeta_2\lambda} \\ \frac{\beta\lambda\zeta_3\omega_x}{\omega_x+\zeta_2\lambda} & \frac{\beta\alpha\omega_x}{\omega_x+\zeta_2\lambda} \end{bmatrix} \begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} + \begin{bmatrix} \frac{\beta^{-1}\zeta_1\omega_x}{\omega_x+\zeta_2\lambda} & -\frac{\zeta_2\beta(1-\alpha)\theta^2}{\omega_x+\zeta_2\lambda} \\ \frac{\beta^{-1}\zeta_1\lambda\omega_x}{\omega_x+\zeta_2\lambda} & \frac{\beta(1-\alpha)\theta^2\omega_x}{\omega_x+\zeta_2\lambda} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix}, \tag{B.12}$$

or

$$\mathbf{y}_t = \mathcal{T}_A(\zeta_2,\zeta_3)E_t\mathbf{y}_{t+1} + \mathcal{T}_B(\zeta_1,\zeta_2)\mathbf{y}_{t-1}. \tag{B.13}$$

Then, by using a condition analogous to (B.7), it is easy to verify that consistency between the ALM and PLM is not possible. Doubts about variables included are justified and therefore the policy is *time-inconsistent*.